

Quantumness of bound entangled states

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Abstract. We employ geometric discord and measurement induced nonlocality to quantify quantumness of some well-known bipartite bound entangled states, namely the two families of Horodecki's ($2 \otimes 4$ and $3 \otimes 3$ dimensional) bound entangled states and that of Bennett *et al*'s in $3 \otimes 3$ dimension. In most of the cases our results are analytic and both the measures attain relatively small value. We found that while viewed as $2 \otimes 8$ system, not only the notion, but also the amount of quantumness of the $4 \otimes 4$ Werner and isotropic states remains the same. The bound entangled state of Benatti *et al* exhibits the same property.

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1. Introduction

Entanglement is the most mysterious feature of quantum mechanics since its early days. In the last two decades, with the emergence of Quantum Information Theory, entanglement has been recognized as a resource. There are two kinds of entanglements, the *free* entanglement which can be used for various information processing tasks like teleportation, super dense coding etc., and the *bound* entanglement which needs a finite amount of free entanglement for its creation, but once created, no free entanglement can be extracted (distilled) from it [1]. The latter one is the most peculiar in the sense that in contrast to its counterpart, it can not be used directly (alone) but when supplied with other resources, can enhance some of the tasks [2]. It is indeed very difficult to detect and even more difficult to quantify bound entanglement. To the best of our knowledge, only two works have quantified entanglement in some well known classes of bound entangled states, one pertaining to bipartite states [3], and the other to multipartite states [4].

Until a few years ago, entanglement was the only *hallmark* of *quantumness* i.e., absence of entanglement and *classicality* were the same notions. With the work of Ollivier and Zurek [5], the notion of quantumness has been extended far beyond entanglement and several measures of nonclassicality have been proposed. Of them, the two interesting ones are the geometric discord (GD) and the measurement-induced nonlocality (MIN). In this work we evaluate these measures for some well-known bound entangled states. We note that the negativity [6], the most common measure of mixed state entanglement, can not quantify quantumness of these states.

The concept of GD was introduced by Dakić *et al* [7] and for an $m \otimes n$ ($m \leq n$) state ρ it is defined as (normalized such that the maximum is 1 [8])

$$\mathcal{D}(\rho) = \frac{m}{m-1} \min_{\chi \in \Omega_0} \|\rho - \chi\|^2 \quad (1)$$

where Ω_0 is the set of zero-discord states (i.e., *classical-quantum states*, given by $\sum p_k |\psi_k\rangle\langle\psi_k| \otimes \rho_k$) and $\|A\|^2 = \text{Tr}(A^\dagger A)$ is the Frobenius or Hilbert-Schmidt norm. Later on Luo and Fu [9] gave an alternative description of GD via a minimization over all possible von Neumann measurements $\Pi^A = \{\Pi_k^A\}$ on ρ^A ,

$$\mathcal{D}(\rho) = \frac{m}{m-1} \min_{\Pi^A} \|\rho - \Pi^A(\rho)\|^2 \quad (2)$$

where $\Pi^A(\rho) := \sum_k (\Pi_k^A \otimes I^B) \rho (\Pi_k^A \otimes I^B)$. Very recently GD has been demonstrated as a better resource than entanglement for remote state preparation [10].

Consider ρ to have the following Bloch form

$$\rho = \frac{1}{mn} \left[I_m \otimes I_n + \mathbf{x}^t \lambda \otimes I_n + I_m \otimes \mathbf{y}^t \lambda + \sum T_{ij} \lambda_i \otimes \lambda_j \right] \quad (3)$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{d^2-1})^t$ with λ_i being the generators of $SU(d)$ for appropriate dimension $d = m, n$ [11]. Then a lower bound on GD is given by [12, 13]

$$\mathcal{D}(\rho) \geq \frac{2}{m(m-1)n} \left[\|x\|^2 + \frac{2}{n} \|T\|^2 - \sum_{k=1}^{m-1} \lambda_k^\downarrow(G) \right] \quad (4)$$

where $\lambda_k^\downarrow(G)$ are the eigenvalues of $G := xx^t + \frac{2}{n}TT^t$ sorted in non-increasing order. The bound is saturated for all $2 \otimes n$ states and in this case the optimal measurements are given by $\Pi_{1,2} = \frac{1}{2}(\mathbf{I} \pm \mathbf{e} \cdot \lambda)$, \mathbf{e} being the eigenvector corresponding to the eigenvalue λ_1^\downarrow . Similarly, for $3 \otimes 3$ systems, if the following operators

$$\Pi_{1,2} = \frac{1}{3}I + \frac{1}{2} \left(\pm \mathbf{e}_1 + \frac{1}{\sqrt{3}}\mathbf{e}_2 \right) \lambda, \quad \Pi_3 = I - \Pi_1 - \Pi_2 \quad (5)$$

are legitimate orthonormal projectors, they constitute the optimal (not necessarily unique) von Neumann measurement and equality holds in Eq. (4).

The other measure of quantumness, namely the MIN is somewhat dual to GD and is defined as (normalized)[‡] [14]

$$\mathcal{M}(\rho) = \frac{m}{m-1} \max_{\Pi^A} \|\rho - \Pi^A(\rho)\|^2 \quad (6)$$

where the maximization is over all von Neumann measurements Π^A which *do not disturb* ρ^A locally i.e., $\sum_k \Pi_k^A \rho^A \Pi_k^A = \rho^A$. The MIN is the finest measure of quantumness till date (though an operational interpretation is not yet known) as it can be non-zero even for a zero-discord state. If ρ^A is non-degenerate, its eigen projectors correspond to the unique optimal von Neumann measurement, otherwise we have the following upperbound on MIN [14, 12]

$$\mathcal{M}(\rho) \leq \frac{4}{m(m-1)n^2} \sum_{k=1}^{m^2-m} \lambda_k^\downarrow(TT^t) \quad (7)$$

Similar to the case of GD, this bound is saturated for all $2 \otimes n$ states. The two bounds given by Eq. (4) and Eq. (7) are the main tools for our analytical study. In other situations, we shall resort to numerical techniques.

2. Horodecki's bound entangled states

2.1. P. Horodecki's $2 \otimes 4$ bound entangled state [15]

The first bound entangled state in the literature (and also in the least possible dimension) is given by

$$\rho_a = \frac{7a}{7a+1} \rho_{\text{ent}} + \frac{1}{7a+1} |\phi\rangle\langle\phi|, \quad a \in [0, 1] \quad (8)$$

where

$$\begin{aligned} |\phi\rangle &= |1\rangle \otimes \left(\sqrt{\frac{1+a}{2}} |0\rangle + \sqrt{\frac{1-a}{2}} |2\rangle \right) \\ \rho_{\text{ent}} &= \frac{2}{7} \sum_{i=1}^3 |\psi_i\rangle\langle\psi_i| + \frac{1}{7} |03\rangle\langle 03| \\ |\psi_i\rangle &= \frac{1}{\sqrt{2}} (|0i-1\rangle + |1i\rangle), \quad i = 1, 2, 3 \end{aligned}$$

[‡] To avoid notational conflict with a well known entanglement measure, negativity [6], we use \mathcal{M} instead of \mathcal{N} .

The state ρ_a remains PPT throughout $a \in [0, 1]$ and for $0 < a < 1$, it is bound entangled. Expressing in Bloch form, we have $x = (0, 0, \frac{a-1}{7a+1})^t$ and the eigenvalues of G are $\frac{24a^2}{(1+7a)^2}$ (multiplicity 2) and $\frac{4(1-a)}{(1+7a)^2}$. Since for $m = 2$, equality holds in the formula (4), we have

$$\mathcal{D}(\rho_a) = \begin{cases} \frac{12a^2}{(1+7a)^2}, & \text{if } a \in [0, \frac{1}{3}] \\ \frac{1+a(6a-1)}{(1+7a)^2}, & \text{if } a \in [\frac{1}{3}, 1] \end{cases} \quad (9)$$

The corresponding optimal von Neumann projectors are given by

$$\Pi_{1,2} = \begin{cases} \frac{1}{2}(\mathbf{I} \pm \sigma_z), & \text{if } a \in [0, \frac{1}{3}] \\ \frac{1}{2}(\mathbf{I} \pm \sigma_x) \text{ or } \frac{1}{2}(\mathbf{I} \pm \sigma_y), & \text{if } a \in [\frac{1}{3}, 1] \end{cases}$$

Let us now evaluate MIN. The eigenvalues of ρ^A are $\frac{4a}{1+7a}$ and $\frac{1+3a}{1+7a}$. So ρ_a is non-degenerate unless $a = 1$. In the non-degenerate case, the optimal von Neumann measurement (which corresponds to the eigen projectors of ρ^A) is given by $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$. Therefore, the MIN is exactly the same as GD in the first case in Eq. (9). In the degenerate case, since equality holds in Eq. (7), we have $\mathcal{M} = 3/16$. This matches with the MIN of non-degenerate case, if we put $a = 1$. Thus we have

$$\mathcal{M}(\rho_a) = \frac{12a^2}{(1+7a)^2} \quad (10)$$

and the optimal measurement is $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$.

We note that for $a = 0$, ρ_a becomes a product state and has a vanishing \mathcal{D} and \mathcal{M} . However, for $a = 1$, though separable, ρ_a has non-vanishing \mathcal{D} and \mathcal{M} .

We also note that the maximum value of \mathcal{D} is $3/25$ (occurs at $a = 1/3$) and the maximum value of \mathcal{M} is $3/16$ (occurs at $a = 1$) which are quite small compared to the maximum possible value 1.

2.2. Horodeckis' $3 \otimes 3$ bound entangled state [16, 17]

The state is given by

$$\rho_\beta = \frac{2}{7}|\Phi\rangle\langle\Phi| + \frac{\beta}{7}\sigma_+ + \frac{5-\beta}{7}\sigma_-, \quad \beta \in [0, 5] \quad (11)$$

where

$$\begin{aligned} |\Phi\rangle &= \frac{1}{\sqrt{3}} \sum_{k=0}^2 |kk\rangle \\ \sigma_+ &= \frac{1}{3} (|01\rangle\langle 01| + |12\rangle\langle 12| + |20\rangle\langle 20|) \\ \sigma_- &= \frac{1}{3} (|10\rangle\langle 10| + |21\rangle\langle 21| + |02\rangle\langle 02|) \end{aligned}$$

This is an interesting class as it contains (depending on the allowed range of the defining parameter β) all kinds of states, namely, negative partial transpose (NPT),

positive partial transpose (PPT), separable, bound and free entangled. Surprisingly, it turns out that this class is also a good example for demonstrating duality between \mathcal{D} and \mathcal{M} .

For this state explicit calculation shows that $x=0$ and the eigenvalues of G are $\frac{6}{49}$ (multiplicity 6) and $\frac{3}{98}(19 - 15\beta + 3\beta^2)$ (multiplicity 2). Accordingly, we have the following bound

$$\mathcal{D}(\rho_\beta) \geq \begin{cases} \frac{1}{49} (9 - 5\beta + \beta^2), & \text{if } \beta \in [\frac{5-\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}] \\ \frac{4}{49}, & \text{otherwise} \end{cases} \quad (12)$$

In the first case, the operators in (5) fail to be orthonormal projectors and so we can not guarantee the equality. Indeed, the best we achieved (which is not by systematic procedure, but by hit and trial) is the value $\frac{1}{294}(49 - 25\beta + 5\beta^2)$, corresponding to the measurement $\Pi_k = |p_k\rangle\langle p_k|$, $k = 1, 2$, with $p_1 = (1, 1, 1)/\sqrt{3}$, $p_2 = (1, 1, -2)/\sqrt{6}$. Thus, for $\beta \in [\frac{5-\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}]$, we have

$$\frac{1}{49} (9 - 5\beta + \beta^2) \leq \mathcal{D}(\rho_\beta) \leq \frac{1}{294} (49 - 25\beta + 5\beta^2) \quad (13)$$

In the second case, however, the operators in (5) are orthonormal projectors ($\Pi_k = |k\rangle\langle k|$) and hence we have the exact value of GD.

Now to evaluate MIN, we note that ρ^A is degenerate (since $x = 0$). So using Eq. (7), we have the following upperbound on MIN

$$\mathcal{M}(\rho_\beta) \leq \begin{cases} \frac{4}{49}, & \text{if } \beta \in [\frac{5-\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}] \\ \frac{1}{49} (9 - 5\beta + \beta^2), & \text{otherwise} \end{cases} \quad (14)$$

Opposite to the case of GD, here equality holds in the first case while for the second case we have

$$\frac{1}{294} (49 - 25\beta + 5\beta^2) \leq \mathcal{M}(\rho_\beta) \leq \frac{1}{49} (9 - 5\beta + \beta^2) \quad (15)$$

We note that Eqs. (12)-(15) clearly demonstrate the duality between GD and MIN. We also note that $\mathcal{D}_{\max} = 4/49$, $\mathcal{D}_{\min} \geq 11/196$ and $\mathcal{M}_{\min} = 4/49$, $\mathcal{M}_{\max} \leq 1/6$. Thus both GD and MIN are small, but always strictly positive, irrespective of entanglement. A comparison between entanglement, GD and MIN has been illustrated in FIG-1.

3. Unextendible-product-basis (UPB) based bound entangled states

3.1. Bennett et al 's $3 \otimes 3$ bound entangled state (Pyramid) [18]

The first UPB based bound entangled state is given by

$$\rho_{Pyramid} = \frac{1}{4} \left(\mathbf{I}_3 \otimes \mathbf{I}_3 - \sum_{i=0}^4 |\psi_i\rangle\langle\psi_i| \right) \quad (16)$$

where $|\psi_i\rangle = |v_i\rangle|v_{2i \bmod 5}\rangle$; $v_i = N(\cos \frac{2\pi i}{5}, \sin \frac{2\pi i}{5}, h)$, $N = 2/(5 + \sqrt{5})^{1/2}$, $h = (1 + \sqrt{5})^{1/2}/2$.

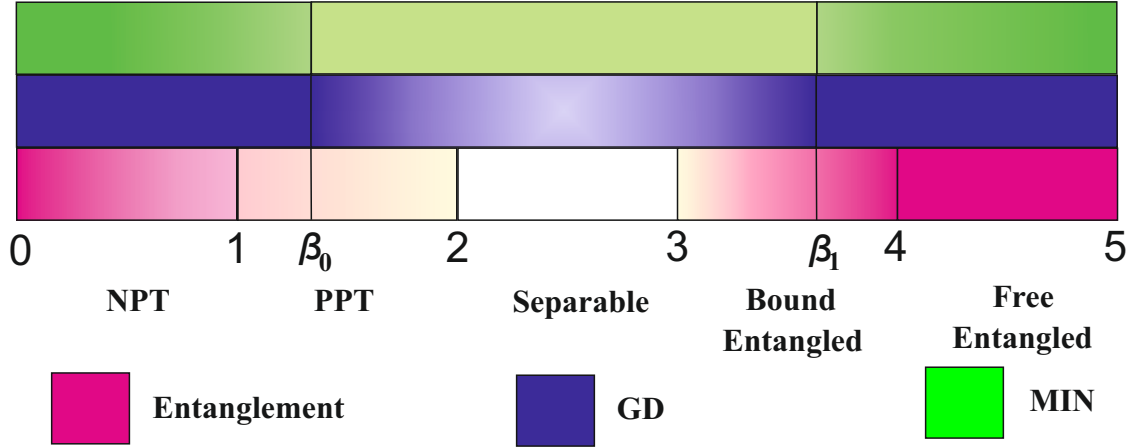


Figure 1. (Color online) Quantumness of the state ρ_β given by Eq. (11). For $0 \leq \beta < 1$, the state is NPT and hence entangled (but it is not known whether free or bound), for $1 \leq \beta \leq 4$ the state is PPT (but unknown whether separable or entangled in $1 \leq \beta < 2$), for $2 \leq \beta \leq 3$ separable, for $3 < \beta \leq 4$ bound entangled and for $4 < \beta \leq 5$ it is free entangled [17]. The state has varying GD within $\beta_0 < \beta < \beta_1$ (\mathcal{D} decreases from β_0 to $5/2$ and then increases symmetrically) but constant outside. Contrary to GD, MIN is fixed within $\beta_0 < \beta < \beta_1$ and varies outside (\mathcal{M} decreases from 0 to β_0 , then remains fixed up to β_1 and increases symmetrically from β_1 to 5). This demonstrates duality between GD and MIN. Note that both \mathcal{D} and \mathcal{M} are always strictly positive irrespective of entanglement. The points $\beta_{0,1}$ are $(5 \mp \sqrt{5})/2$.

Using Eq. (4) we have the following bound on GD

$$\mathcal{D} \geq \frac{1}{32} (19 - 7\sqrt{5}) = 0.10461 \text{ (up to five decimal places).}$$

It can be easily seen that the operators in Eq. (5) fail to be projectors, leading to difficulties in analytic calculations. However we note that these operators originated from a particular choice of unitary and there may be other measurements (corresponding to different choice of unitary) which would saturate the bound [12]. Indeed it turns out that the projection operators $\Pi_k = |p_k\rangle\langle p_k|$, $k = 1, 2$ with

$$|p_1\rangle = \frac{1}{\sqrt{6}} (\sqrt{3}, 1, \sqrt{2}), \quad |p_2\rangle = \frac{1}{\sqrt{3}} (0, -\sqrt{2}, 1)$$

saturate this bound and hence the exact GD is given by

$$\mathcal{D} = \frac{1}{32} (19 - 7\sqrt{5}) = 0.10461 \quad (17)$$

Unfortunately, ρ^A is degenerate and hence using the bound of Eq. (7), we get

$$\mathcal{M} \leq \frac{5\sqrt{5}}{48} (3 - \sqrt{5}) = 0.17794$$

As in the case of GD, this bound is not saturated by the operators in Eq. (5). But with a careful analysis, we can get the analytic MIN in this case. First we note that the non degenerate eigenvector of ρ^A is $|2\rangle\langle 2|$. In order to keep ρ^A invariant, the other two projectors must be from $\text{span}\{|0\rangle, |1\rangle\}$ only and so, without loss of generality, can

be taken as $\Pi_{1,2} = (1/2)(I \pm x \cdot \sigma)$ with $x = (a, b, c) \in \mathbb{R}^3, |x| = 1$. After some algebraic simplification, this measurement gives

$$\|\rho - \Pi^A(\rho)\|^2 = \frac{1}{8} (\sqrt{5} - 2) (3 + b^2)$$

Clearly the maximum occurs at $b = 1$, thereby we get the exact MIN as

$$\mathcal{M} = \frac{3}{4} (\sqrt{5} - 2) = 0.17705 \quad (18)$$

and the optimal von Neumann projectors are given by

$$\Pi_{1,2} = \frac{1}{2}(I \pm \sigma_y), \quad \Pi_3 = |2\rangle\langle 2|.$$

3.2. Bennett et al 's $3 \otimes 3$ bound entangled state (Tiles) [18]

Another UPB-based bound entangled state from the same work is

$$\rho_{\text{Tiles}} = \frac{1}{4} \left(\mathbf{I}_3 \otimes \mathbf{I}_3 - \sum_{i=0}^4 |\psi_i\rangle\langle\psi_i| \right) \quad (19)$$

where $|\psi_0\rangle = |0\rangle(|0\rangle - |1\rangle)/\sqrt{2}$, $|\psi_1\rangle = (|0\rangle - |1\rangle)|2\rangle/\sqrt{2}$, $|\psi_2\rangle = |2\rangle(|1\rangle - |2\rangle)/\sqrt{2}$, $|\psi_3\rangle = (|1\rangle - |2\rangle)|0\rangle/\sqrt{2}$, $|\psi_4\rangle = (|0\rangle + |1\rangle + |2\rangle)(|0\rangle + |1\rangle + |2\rangle)/3$.

For this state, analytic evaluation of GD is very complicated. Even calculating the bound in Eq. (4) is quite difficult. For example, the eigenvalues of G are $(19 + 2r \cos \theta)/64$, $27/64$ (multiplicity 2), $(19 - 2r \cos(\theta \pm \pi/3))/64$ and 0 (multiplicity 3), where $r = \sqrt{55}$ and $\theta = (1/3) \arctan(9\sqrt{1319}/244)$. Thus we have the following bound

$$\begin{aligned} \mathcal{D} &\geq \frac{1}{576} \left(65 - 2\sqrt{55} \cos \left[\frac{1}{3} \arctan \left(\frac{9\sqrt{1319}}{244} \right) \right] \right) \\ &= 0.08832 \end{aligned}$$

We use numerical techniques to judge this bound. A simulation of 10^5 measurements has been shown in FIG-2. Numerically we have achieved the value 0.088990. We note that the bound is quite good as it gives correct result up to two decimal places.

Fortunately, however, ρ^A is non degenerate. Hence we have the exact value of MIN. The (unnormalized) eigen projectors of ρ^A are $(1, (5 - \sqrt{33})/2, 1)$, $(-1, 0, 1)$ and $(1, (5 + \sqrt{33})/2, 1)$. Since they correspond to the optimal measurement, we have

$$\mathcal{M} = \frac{95}{704} = 0.13494 \quad (20)$$

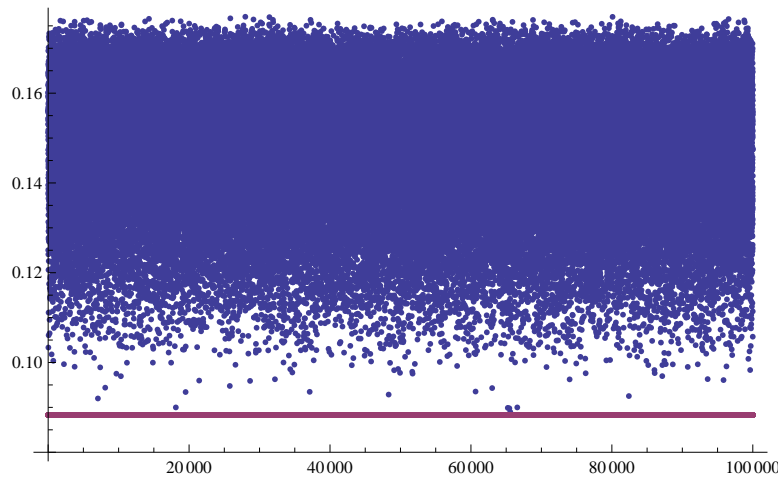


Figure 2. (Color online) A simulation of 10^5 random measurements on the state in Eq. (19). The red solid line represents the bound of $\mathcal{D} \geq 0.08832$, whereas we have achieved the value 0.088990.

4. Benatti *et al* 's $4 \otimes 4$ bound entangled state [19]

The state ρ has the following matrix form in computational basis (the zero entries are not written to retain clarity of the nice pattern)

$$\rho = \frac{1}{24} \begin{pmatrix} 1 & & & -1 & & -1 & & & 1 \\ & 3 & & -1 & & & & -1 & -1 \\ & & 1 & & & -1 & -1 & & 1 \\ & & & 1 & & 1 & & 1 & \\ -1 & & & & 3 & & & -1 & -1 \\ -1 & -1 & & 1 & & & 1 & & -1 \\ & & 1 & & 1 & & 1 & & 1 \\ & & & -1 & & 1 & 1 & & -1 \\ & & -1 & & & 1 & 1 & & -1 \\ & & & 1 & & 1 & & 1 & 1 \\ -1 & & & 1 & & & 1 & & -1 \\ & -1 & & -1 & & & & 3 & -1 \\ & & 1 & & 1 & & 1 & & 1 \\ & & & 1 & & -1 & -1 & & 1 \\ & -1 & & -1 & & & & -1 & 3 \\ 1 & & & & -1 & & -1 & & 1 \end{pmatrix}$$

This state arises in the study of non decomposable positive maps and it turns out that it has some similarities with the Werner and isotropic states. For example, it has maximally mixed subsystems and, as we are going to show, the amount of GD coincides with the amount of MIN and the optimal measurement is in the computational basis.

To evaluate GD, we note that $x = 0$ and $TT^t = (4/9)I_{15}$. The corresponding operators of Eq. (5) are the computational basis $\{|k\rangle\langle k|\}$ and hence we have the exact

value of GD that is $\mathcal{D} = 1/9$. As ρ^A is fully degenerate, finding MIN in the general case is very difficult. However, we note that if $x = 0$ and the eigenvalues of TT^t are equal, the two bounds in Eq. (4) and Eq. (7) coincide and hence if one of them is saturated, the other follows immediately, with the same measurement (this was exactly the case in Werner and isotropic states [12]). Thus we have $\mathcal{D} = \mathcal{M} = 1/9$.

Another interesting property of this state is that it remains bound entangled even if we view it (the matrix representation) as the density matrix of a $2 \otimes 8$ system [20] (in general, the same matrix may represent a separable or entangled system depending on dimension of the subsystems involved [21]). By a tedious but somewhat straightforward calculation it follows that $x = 0$ (though $y \neq 0$) and $TT^t = (4/9)I_3$. Hence for this $2 \otimes 8$ case, $\mathcal{D} = \mathcal{M} = 1/9$ holds too. Out of curiosity, we have checked that the same holds for the Werner and isotropic states, namely, when seen as $2 \otimes 8$ systems they both have $x = 0, y \neq 0$, $\mathcal{D} = \mathcal{M}$ and even this value (of $\mathcal{D} = \mathcal{M}$) does not change from the $4 \otimes 4$ case.

5. Discussion and conclusion

The bound entangled states appeared more than a decade back and have been studied extensively from several perspectives. The notion of geometric discord and measurement induced nonlocality has been introduced very recently and since then these measures have been evaluated for many classes of states. In this work for the first time we evaluated these measures for bound entangled states. We found that both the measures attain relatively very small value. In some sense this shows that these states are *weakly* quantum. Besides the states presented here, we have studied some other bound entangled states as well; but we always got small values for these measures. It would be interesting to find bound entangled states where these measures attain the values close to one. If there are no such states, then it will be a challenging problem to find the maximum value. We hope this work will generate interest to explore these issues which may lead to a better understanding of bound entangled states.

As mentioned earlier, the evaluation of measurement induced nonlocality becomes very difficult when ρ^A has degenerate spectrum. In Sec. 3.1, we have shown how the case of small degeneracy could be dealt. We hope this approach can be generalized to arbitrary states which will make the evaluation quite easy.

To conclude, we have evaluated two post-entanglement measures of quantumness, namely geometric discord and measurement induced nonlocality for various bound entangled states. We have given analytic results in most of the cases. In particular, duality between the two measures has been reflected well in the Horodecki's state ρ_β given in Eq. (11). When viewed as $2 \otimes 8$ system, similar to the Werner and isotropic states, not only the notion but also the amount of quantumness of Benatti *et al*'s $4 \otimes 4$ bound entangled state remains the same.

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